13.4. Motion in space

Ex A particle was fired at $(0,0,6)$ with initial velocity $\vec{v_0} = (3,4,1)$, and moves with constant acceleration $\vec{a} = (0,0,-2)$. Find the position where it hits the ground $z=0$.

Sol $\vec{r}(t)$: position at time $t \geq 0$.

$$\Rightarrow \begin{cases} \vec{r}(0) = (0,0,6) \\ \vec{r}'(0) = \vec{v_0} = (3,4,1) \\ \vec{r}''(t) = \vec{a} = (0,0,-2) \end{cases}$$

$$\vec{r}'(t) = \vec{r}'(0) + \int_0^t \vec{r}''(u)\,du$$

Fund. Thm

$$= (3,4,1) + \int_0^t (0,0,-2)\,du$$

$$= (3,4,1) + \left(\int_0^t 0\,du,\ \int_0^t 0\,du,\ \int_0^t -2\,du \right)$$

$$= (3,4,1) + (0,0,-2t)$$

$$= (3,4,1-2t).$$

$$\vec{r}(t) \underset{\substack{\uparrow \\ \text{Fund. Thm}}}{=} \vec{r}(0) + \int_0^t \vec{r}\,'(u)\,du$$

$$= (0,0,6) + \int_0^t (3, 4, 1-2u)\,du$$

$$= (0,0,6) + \left(\int_0^t 3\,du,\ \int_0^t 4\,du,\ \int_0^t 1-2u\,du\right)$$

$$= (0,0,6) + (3t, 4t, t-t^2)$$

$$= (3t, 4t, 6+t-t^2).$$

$$\underbrace{z=0}_{\text{ground}} \Rightarrow 6+t-t^2=0 \underset{\substack{\uparrow \\ \text{factorization or quadratic formula}}}{\Rightarrow} t=3, \cancel{-2} \quad t \geq 0$$

The particle hits the ground at

$$\vec{r}(3) = (3\cdot 3, 4\cdot 3, 6+3-3^2)$$

$$= \boxed{(9, 12, 0)}$$

<u>Note</u> You are expected to know how to solve a quadratic equation using a factorization or the quadratic formula.

Ex Consider two moving particles whose positions at time $t \geq 0$ are given by

$$\vec{r_1}(t) = (t+1,\ 2t-1,\ 3t-4)$$

$$\vec{r_2}(t) = (3t,\ 2t^2+1,\ -t^2+3)$$

(1) Determine whether they collide.

Sol They collide when they are at the same position at the same time.

$\rightsquigarrow \vec{r_1}(t) = \vec{r_2}(t)$

$\rightsquigarrow (t+1,\ 2t-1,\ 3t-4) = (3t,\ 2t^2+1,\ -t^2+3)$

$\rightsquigarrow t+1 = 3t,\quad 2t-1 = 2t^2+1,\quad 3t-4 = -t^2+3.$

1^{st} equation : $t+1 = 3t \Rightarrow t = \frac{1}{2}.$

But $t = \frac{1}{2}$ does not work for all equations.

$\left(t = \frac{1}{2} \Rightarrow 2t-1 = 0,\ 2t^2+1 = \frac{3}{2} \Rightarrow 2t-1 \neq 2t^2+1\right)$

$\Rightarrow$ no solutions for t

$\Rightarrow$ The particles $\boxed{\text{do not collide}}$

(2) Find all intersection points of their paths.

Sol Their paths intersect if they reach the same point at possibly different times.

$\rightsquigarrow \vec{r}_1(t) = \vec{r}_2(u)$

$\rightsquigarrow (t+1,\ 2t-1,\ 3t-4) = (3u,\ 2u^2+1,\ -u^2+3)$

$\rightsquigarrow t+1 = 3u,\ 2t-1 = 2u^2+1,\ 3t-4 = -u^2+3.$

1st equation : $t+1 = 3u \Rightarrow t = \underline{3u-1}.$

2nd equation : $2(3u-1)-1 = 2u^2+1$

$\rightsquigarrow 2u^2 - 6u + 4 = 0 \Rightarrow u = 1, 2$

factorization or quadratic formula

$u=1 \Rightarrow t = 3u-1 = 2,$

$u=2 \Rightarrow t = 3u-1 = 5.$

Check 3rd equation.

$\begin{cases} u=1, t=2 \Rightarrow 3t-4 = 2,\ -u^2+3 = 2 \\ u=2, t=5 \Rightarrow 3t-4 = 11,\ -u^2+3 = -1. \end{cases}$

$\Rightarrow u=1, t=2.$

Their paths intersect at

$\vec{r}_1(2) = \vec{r}_2(3) = \boxed{(3, 3, 2)}$

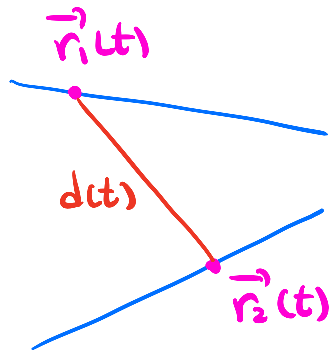
Ex Consider two particles whose positions at time t are given by

$$\vec{r_1}(t) = (3t,\ 3+4t,\ 1+2t)$$

$$\vec{r_2}(t) = (3+2t,\ 2+3t,\ 5+2t)$$

(1) Find the minimum distance between the two particles.

Sol At time t, their distance is



$$\begin{aligned} d(t) &= |\vec{r_2}(t) - \vec{r_1}(t)| \\ &= |(3-t,\ -1-t,\ 4)| \\ &= \sqrt{(3-t)^2 + (-1-t)^2 + 4^2} \\ &= \sqrt{2t^2 - 4t + 26} \end{aligned}$$

$\Rightarrow d(t)^2 = 2t^2 - 4t + 26 = 2(t-1)^2 + 24$

$d(t)^2$ attains the minimum 24 at $t=1$.

$\Rightarrow d(t)$ attains the minimum $\sqrt{24}$ at $t=1$.

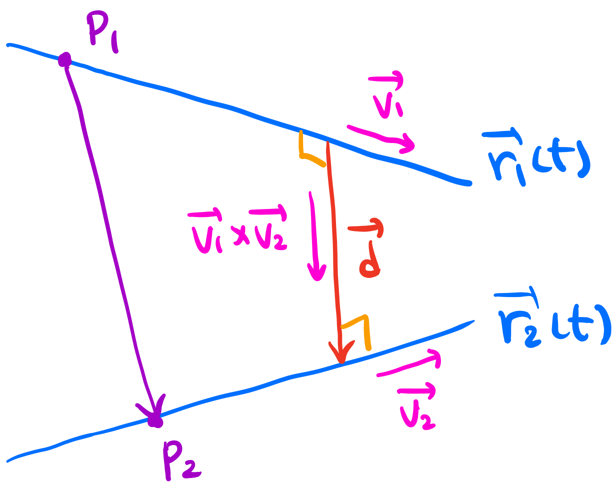
$\Rightarrow$ The minimum distance is $\boxed{\sqrt{24}}$

Note You can also find the minimum of $d(t)$ by finding the critical points.

(2) Find the shortest distance between their trajectories.

<u>Sol</u> Both $\vec{r_1}(t)$ and $\vec{r_2}(t)$ parametrize lines with direction vectors

$$\vec{v_1} = (3,4,2) \quad \text{and} \quad \vec{v_2} = (2,3,2).$$



The shortest distance between the two lines is equal to $|\vec{d}|$.

We find the vector $\vec{d}$ that is perpendicular to both lines and connects them.

$\rightsquigarrow$ $\vec{d}$ is parallel to $\vec{v_1} \times \vec{v_2} = (2,-2,1)$.

Choose a point on each line:

$$\begin{cases} P_1 = \vec{r_1}(0) = (0,3,1) \\ P_2 = \vec{r_2}(0) = (3,2,5) \end{cases}$$

$\rightsquigarrow$ $\overrightarrow{P_1P_2} = (3,-1,4).$

Then $\vec{d} = \text{Proj}_{\vec{v_1} \times \vec{v_2}} \overrightarrow{P_1P_2}$

$$= \frac{\overrightarrow{P_1P_2} \cdot (\vec{v_1} \times \vec{v_2})}{|\vec{v_1} \times \vec{v_2}|^2} \vec{v_1} \times \vec{v_2}$$

$\uparrow$ projection formula

$$\overrightarrow{P_1P_2} \cdot (\vec{v_1} \times \vec{v_2}) = (3, -1, 4) \cdot (2, -2, 1) = 12$$

$$|\vec{v_1} \times \vec{v_2}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

$$\Rightarrow \vec{d} = \frac{12}{3^2} (2, -2, 1) = \frac{4}{3} (2, -2, 1)$$

The shortest distance between the two lines is given by

$$|\vec{d}| = \sqrt{\frac{4}{3} (2, -2, 1)} = \frac{4}{3} \sqrt{2^2 + (-2)^2 + 1^2}$$

$$= \frac{4}{3} \cdot 3 = \boxed{4}$$